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A four-step trigonometric fitted P-stable Obrechkoff method for periodic initial-value problems

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Abstract

In this paper, we present a new P-stable Obrechkoff four-step method, which greatly improves the performance of our previous Obrechkoff four-step method and extends its application range. By trigonometric fitting, we extend the interval of periodicity of the previous four-step method from about $H^2 \sim 16$ to infinity and at the same time, we keep all its advantage in the accuracy and efficiency. We have tested the new method by four well-known problems, (1) the test-equation; (2) Stiefel and Bettis problem; (3) Duffing equation without damping; and (4) Bessel equation. The numerical results show that the new method is more accurate than any previous method. It also has great advantage in stability and efficiency.

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Keywords: Obrechkoff method; P-stable; High-order derivative; First-order derivative formula; Second-order initial value problem with periodic solutions

1. Introduction

In this paper, we will consider the second-order ordinary differential equation (ODE) for the initial-value problems with the periodic solutions, which can be written as

$$y''(x) = f(x, y(x), y'(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{1}$$

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For several decades, there has been a strong interest in searching for a better numerical method to integrate this equation [1,2,4–17], because it has many important applications such as to find an eigenvalue in Schrödinger equation, to find solutions to the Bessel equation with various boundary conditions, and to study the chaos behavior from Duffing equation. Using higher order derivative to improve the accuracy and extend the stability region, finding a high accurate and high efficient Obrechkoff method has become an important research field in the numerical method [2,5,7,15–17].

Recently, we published an Obrechkoff four-step method with a special structure [3]. From the numerical test, it is shown that this method has great advantage in accuracy and efficiency over many other methods. However, this method has a fatal disadvantage of small interval of periodicity (about 16 for the widest). This disadvantage restricts its application in some area where a large interval of periodicity is required. In this paper we want to improve our four-step method for its stability.

The purpose of this paper is to continue our previous work in Obrechkoff multi-step method. We want to use trigonometric fitting to extend the interval of periodicity of our previous method to infinity and make it P-stable. This is the first to report a P-stable four-step Obrechkoff method, as we know. This paper is arranged as follows: in the next section, we give the derivation of the new method, in Section 3, we will test our new method with four well-known problems and in the last section, we give a conclusion.

2. Derivation

In this section, we will derive our new method and verify whether our new four-step Obrechkoff method is a P-stable one.

To build our P-stable method, we begin with the following structure, the same as that in [3],

$$\begin{aligned} &y_{n+2} + y_{n-2} + a(y_{n+1} + y_{n-1}) - (2+2a)y_n \\ &\quad = -h^2[A_2(y''_{n+2} + y''_{n-2}) + A_1(y''_{n+1} + y''_{n-1}) + A_0 y''_n] \\ &\qquad - h^4[B_1(y^{(4)}_{n+1} + y^{(4)}_{n-1}) + B_0 y^{(4)}_n] - h^6[b(y^{(6)}_{n+1} + y^{(6)}_{n-1}) + C_0 y^{(6)}_n], \end{aligned} \tag{2}$$

where two special coefficients a and b will be determined by the requirement of P-stable.

We firstly use the Taylor series expansion to determine all the coefficients of (2) except a and b, which can be written as

$$\begin{aligned} &\{A_0, A_1, A_2, B_1, B_0, C_0\} \\ &\quad = \left\{ -\frac{8\,214\,784 + 1\,956\,907a + 2\,740\,288\,320b}{2\,069\,760}, \frac{128 - 661a + 16\,133\,040b}{24\,255}, \right. \\ &\qquad -\frac{258\,304 + 127a + 39\,251\,520b}{12\,418\,560}, -\frac{20\,992 - 65a + 5\,322\,240b}{194\,040}, \\ &\qquad \left. -\frac{3\,225\,856 + 176\,065a + 1\,855\,465\,920b}{3\,104\,640}, -\frac{43\,776 + 857a + 26\,685\,120b}{1\,034\,880} \right\}, \end{aligned} \tag{3}$$

and the local truncation error as

$$\mathrm{LTE}(h) = -\frac{496\,576 + 1021a + 211\,351\,140b}{12\,586\,365\,792\,000} h^{14} y_n^{(14)}. \tag{4}$$

Next, we will find the P-stable condition for our new method (2). Let us consider the following test equation:

$$y''(x) = -\omega^2 y(x), \tag{5}$$

with the characteristic roots $y(x) = e^{\pm i\omega x}$.

We assume (2) is a stable method, so $y(x) = e^{\pm i\omega x}$ must be the characteristic roots of (2) too. We substitute $y_n \to e^{i\omega n h}$ and let $e^{in\omega h} \to \lambda^n$ and $H = h\omega$, and obtain the characteristic equation for our new method, which is a symmetric quartic equation in λ:

$$A(H)(\lambda^2 + \lambda^{-2}) + B(H)(\lambda + \lambda^{-1}) + C(H) = 0, \tag{6}$$

where

$$\begin{aligned}
A(H) &= 1 + \frac{1009H^2}{48\,510} + \frac{127H^2 a}{12\,418\,560} + \frac{177H^2 b}{56},\\
B(H) &= -\frac{128H^2}{24\,255} - \frac{2624H^4}{24\,255} + \left(1 + \frac{661H^2}{24\,255} + \frac{13H^4}{38\,808}\right) a - \left(\frac{4656H^2}{7} + \frac{192H^4}{7} + H^6\right),\\
C(H) &= \left(\frac{37\,071H^2}{28} - \frac{8367H^4}{14} + \frac{361H^6}{14}\right) b - 2 + \frac{32\,089H^2}{8085} - \frac{25\,202H^4}{24\,255} + \frac{114H^6}{2695}\\
&\quad - \left(2 - \frac{1\,956\,907H^2}{2\,069\,760} + \frac{35\,213H^4}{620\,928} - \frac{857H^6}{1\,034\,880}\right) a.
\end{aligned} \tag{7}$$

Eq. (6) can be easily solved if we let $z = \frac{1}{2}(\lambda + 1/\lambda)$. Obviously, if $|z| \leqslant 1$, then

$$\lambda_{\pm} = z \pm i\sqrt{1 - z^2} = e^{\pm i\theta}, \tag{8}$$

where $\cos\theta = z$. Hence, $|z| \leqslant 1$ is the condition for stability. After substituting, (6) becomes

$$z^2 + \xi z + \chi = 0, \tag{9}$$

where

$$\xi = \frac{B(H)}{2A(H)}, \quad \chi = \frac{C(H)}{4A(H)} - \frac{1}{2}. \tag{10}$$

The two roots of (9) can be written as

$$z_1 = \frac{1}{2}\left(-\xi - \sqrt{\xi^2 - 4\chi}\right), \quad z_2 = \frac{1}{2}\left(-\xi + \sqrt{\xi^2 - 4\chi}\right). \tag{11}$$

In order to satisfy the condition for stability, we let $|z_1| \leqslant 1$ and $|z_2| \leqslant 1$, both of them, being not > 1, as shown in our previous work [3]. Obviously, this condition can be satisfied only by a small range of H. How can we find a simple way to extend the interval of periodicity to infinity? We find it is easy and let

$$z_1 = \cos(H), \quad z_2 = \sin(H). \tag{12}$$

Obviously, if the condition of (12) can be satisfied, then our new four-step method is P-stable. Therefore, we call (12) the P-stable condition for our four-step method.

We use (12) to determine a and b, which can be written as

$$a = \frac{\rho_2 \cos(H)\sin(H) + \rho_1(\cos(H) + \sin(H)) + \rho_0}{\sigma_2 \cos(H)\sin(H) + \sigma_1(\cos(H) + \sin(H)) + \sigma_0},$$

$$b = \frac{\theta_2 \cos(H)\sin(H) + \theta_1(\cos(H) + \sin(H)) + \theta_0}{\gamma_2 \cos(H)\sin(H) + \gamma_1(\cos(H) + \sin(H)) + \gamma_0}, \tag{13}$$

where

$$\begin{aligned}
\rho_2 &= 8\,260\,116\,480 + H^2(512\,225\,280 + H^2(15\,257\,088 + 258\,304H^2)),\\
\rho_1 &= -8\,260\,116\,480 + H^2(3\,617\,832\,960 - H^2(103\,315\,968 + 2\,500\,096H^2)),\\
\rho_0 &= 8\,260\,116\,480 + H^2(-7\,747\,891\,200 + H^2(2\,256\,403\,968\\
&\quad + H^2(-211\,373\,312 + H^2(8\,284\,416 - 131\,328H^2)))),\\
\sigma_2 &= -39\,251\,520 - H^2(1\,154\,160 + H^2(16\,632 + 127H^2)),\\
\sigma_1 &= 39\,251\,520 + H^2(-18\,471\,600 + H^2(1\,075\,032 - 14\,615H^2)),\\
\sigma_0 &= -39\,251\,520 + H^2(38\,097\,360 - H^2(11\,946\,312\\
&\quad - H^2(1\,376\,297 - H^2(78\,729 - 2571H^2)))),\\
\theta_2 &= -77\,110\,316\,236\,800 - H^2(3\,705\,300\,910\,080 + H^2(69\,543\,936\,000 + 622\,583\,808H^2)),\\
\theta_1 &= 77\,110\,316\,236\,800 - H^2(34\,849\,857\,208\,320 - H^2(1\,429\,823\,324\,160\\
&\quad + H^2(13\,140\,492\,288 - 647\,420\,928H^2))),\\
\theta_0 &= -77\,110\,316\,236\,800 + H^2(73\,405\,015\,326\,720 - H^2(22\,067\,681\,771\,520\\
&\quad - H^2(2\,260\,946\,214\,912 - H^2(102\,673\,342\,464 - 2\,000\,216\,064H^2)))),\\
\gamma_2 &= H^2(243\,723\,678\,105\,600 + H^2(7\,166\,502\,604\,800\\
&\quad + H^2(103\,272\,744\,960 + 788\,578\,560H^2))),\\
\gamma_1 &= -(H^2(243\,723\,678\,105\,600 - H^2(114\,695\,336\,448\,000\\
&\quad - H^2(6\,675\,174\,696\,960 - 90\,748\,627\,200H^2)))),\\
\gamma_0 &= H^2(243\,723\,678\,105\,600 - H^2(236\,557\,175\,500\,800 - H^2(74\,177\,996\,175\,360\\
&\quad - H^2(8\,545\,813\,436\,160 - H^2(488\,850\,405\,120 - 15\,964\,058\,880H^2))))).
\end{aligned} \tag{14}$$

It should be noted that both the numerator and the denominator of (13) may contain zero for a small H. These zeroes in the dominator will destroy the P-stable condition and cause the numerical calculation to become unstable. In order to avoid disaster in the calculation, we need to expand a, b to be a power series of H for the applications with small H.

For some problems, such as in the nonlinear Duffing equation and Bessel equation given in the next section, we need to use a first-order derivative formula to calculate y'. The accuracy of the Obrechkoff method depends on the main structure as well as the first-order derivative formula. We will use two different first-order derivative formulas to calculate Duffing equation and Bessel equation, respectively.

3. Numerical test

In order to test the stability and accuracy of the new method, we apply it to four well-known periodic problems: (1) test equation, (2) Stiefel and Bettis problem, (3) Duffing's equation without damping and Bessel equation.

Problem 1. We consider the problem:

$$y''(x) + \omega^2 y(x) = 0, \quad y(0) = 1, \quad y'(0) = 0, \tag{15}$$

where $\omega^2 = 100$, and the exact solution $y(x) = \cos(\omega x)$.

We integrate (15) numerically by three methods: our new method (2), Simos' method [12] and method (2.11) [2]. The numerical results are shown in Table 1. From the table, it shows that our new method is more accurate than the other methods.

Problem 2. We consider the following 'almost periodic' problem studied by Stiefel and Bettis:

$$z''(x) + z(x) = 0.001e^{ix}, \quad z(0) = 1, \quad z'(0) = 0.9995i, \quad z \in \mathscr{C}, \tag{16}$$

whose theoretical solution represents a motion of the perturbed circular orbit in the complex plane and can be written as

$$z(x) = (1 - i0.0005x)\cos(x) + (i0.9955 + 0.0005x)\sin(x). \tag{17}$$

Solution for z_k was computed with the step size of $h = \pi, \pi/2, \pi/3, \ldots, \pi/6$ in the range of $0 < x < 40\pi$ (which corresponds to 20 orbits of the point $z(x)$). The errors, $|z_{N_k}| - |z(40\,\pi)|$, where $N_k = \mathtt{Int}(40\pi/h)$, are listed in Table 2 for comparison with method (2.11) [2], Simos' method [12] and our new method.

Problem 3. We consider the nonlinear undamped Duffing's equation

$$y''(x) + y(x) + y(x)^3 = B\cos(vx), \tag{18}$$

with $B = 0.002$ and $v = 1.01$. We use the same 'exact' solution as the one in [12]

$$g(x) = \sum_{i=0}^{4} K_{2i+1}\cos[(2i+1)x], \tag{19}$$

where

$$\begin{aligned}\{K_1, K_3, K_5, K_7, K_9\} &= \{0.200179477536266\,91, 2.4694614325577866 \cdot 10^{-4}, 3.040149847079104 \cdot 10^{-7},\\ &\quad 3.743479822186476 \cdot 10^{-10}, 4.584640351544518 \cdot 10^{-13}\}.\end{aligned} \tag{20}$$

Table 1
Absolute errors, $|\Delta y_n| = |y_n - y(nh)|$ for problem (15) are listed with $h = \pi/12$ for comparison among three methods: method (2.11) in [2], Simos's method [12] and our new method (2)

x	(2.11)	Simos	New
π	$2.06 \cdot 10^{-6}$	$3.45 \cdot 10^{-8}$	$2.22 \cdot 10^{-15}$
2π	$9.08 \cdot 10^{-6}$	$1.52 \cdot 10^{-7}$	$1.78 \cdot 10^{-15}$
4π	$3.80 \cdot 10^{-5}$	$6.34 \cdot 10^{-7}$	$7.55 \cdot 10^{-15}$
6π	$8.67 \cdot 10^{-5}$	$1.45 \cdot 10^{-6}$	$8.88 \cdot 10^{-15}$
8π	$1.55 \cdot 10^{-4}$	$2.59 \cdot 10^{-6}$	$9.77 \cdot 10^{-15}$
10π	$2.43 \cdot 10^{-4}$	$4.07 \cdot 10^{-6}$	$4.44 \cdot 10^{-16}$

Table 2
Comparison of the errors in the approximations of z_{N_k}, where $N_k = \mathrm{Int}(40\,\pi/h)$, produced by the method (2.11) in [2], Simos's method [12] and our new method (2) for problem (16) with $h = \pi, \pi/2, \pi/3, \ldots, \pi/6$

h	(2.12)	Simos	New
π	$3.36 \cdot 10^{-1}$	$6.29 \cdot 10^{-1}$	$4.99 \cdot 10^{-3}$
$\pi/2$	$5.32 \cdot 10^{-4}$	$9.92 \cdot 10^{-5}$	$6.54 \cdot 10^{-9}$
$\pi/3$	$4.90 \cdot 10^{-5}$	$2.63 \cdot 10^{-8}$	$3.22 \cdot 10^{-12}$
$\pi/4$	$8.89 \cdot 10^{-6}$	$2.21 \cdot 10^{-10}$	$3.58 \cdot 10^{-11}$
$\pi/5$	$2.35 \cdot 10^{-6}$	$2.67 \cdot 10^{-10}$	$3.88 \cdot 10^{-14}$
$\pi/6$	$7.91 \cdot 10^{-7}$	$4.30 \cdot 10^{-11}$	$4.31 \cdot 10^{-14}$

We briefly outline the algorithm for our P-stable four-step method, which is similar to our previous one, and the detail can be found in [3]. Using (18), we can change (2) into a cubic equation of y_{n+2}, which can be written as

$$y_{n+2}^3 + p y_{n+2} + q = 0, \tag{21}$$

where $p = 1 - 1/(h^2 A_2)$, q is the remaining terms.

Since A_2 is smaller than zero, the cubic equation has two imaginary and one real root. It can easily shown that the real one is just what we need. The iterative formula can be written as

$$y_{n+2} = \left(-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{1/3} - \left(\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^{1/3}. \tag{22}$$

By using (22) to calculate y_{n+2}, we need to know all the values of high-derivatives at other nodes before $n+2$. We can replace a high derivative, such as $y_{n+1}^{(6)}$, by a linear function of y_{n+1} and y'_{n+1} in terms of (18). If y'_{n+1} is known, $y_{n+1}^{(6)}$ can be easily computed. Therefore, we need a first-order derivative formula to calculate y'_{n+1}.

Table 3
Absolute errors, $|\Delta y_n| = |y_n - y(nh)|$, calculated for the problem (18), with $h = \pi/5$, by method (2) are listed, where the numerical results from Hairer [4], (2.12) in [2], Neta's method [5] and Simos' method [12] are all listed for comparison

h	x	Hairer	(2.12)	Neta	Simos	New
$\frac{\pi}{5}$	π	$5.03 \cdot 10^{-2}$	$4.53 \cdot 10^{-5}$	—	$1.85 \cdot 10^{-7}$	$5.76 \cdot 10^{-9}$
	2π	$4.79 \cdot 10^{-2}$	$1.88 \cdot 10^{-4}$	$2.04 \cdot 10^{-5}$	$3.25 \cdot 10^{-6}$	$1.44 \cdot 10^{-8}$
	4π	$4.25 \cdot 10^{-2}$	$7.46 \cdot 10^{-4}$	$8.09 \cdot 10^{-5}$	$7.57 \cdot 10^{-6}$	$2.02 \cdot 10^{-10}$
	6π	$3.64 \cdot 10^{-2}$	$1.63 \cdot 10^{-3}$	$1.80 \cdot 10^{-4}$	—	$1.46 \cdot 10^{-8}$
	8π	$2.97 \cdot 10^{-2}$	$2.78 \cdot 10^{-3}$	$3.15 \cdot 10^{-4}$	$4.57 \cdot 10^{-5}$	$4.78 \cdot 10^{-10}$
	10π	$2.26 \cdot 10^{-2}$	$4.11 \cdot 10^{-3}$	$4.82 \cdot 10^{-4}$	$2.38 \cdot 10^{-4}$	$1.32 \cdot 10^{-8}$

In order to improve the accuracy and efficiency further, we use the following first-order formula,

$$\begin{aligned}
y'_{n+1} &= \frac{960}{L}\left(\Delta_0 - \frac{89h}{320}\left(y'_{n-1} + \frac{h}{267}(\Delta_1 + h\Lambda)\right)\right),\\
L &= h(-267 + h^2(1 + 3y_{n+1}^2)),\\
\Lambda &= Bv\sin(v(n+1)h) + y_{n-1}^{(3)} - 16h\left(y_n^{(4)} + \frac{h^2}{120}y_n^{(6)}\right),\\
\Delta_0 &= y_{n+1} - 2y_n + y_{n-1},\\
\Delta_1 &= 27(y''_{n+1} + y''_{n-1}) - 480y''_n.
\end{aligned} \tag{23}$$

The local truncation error of this first-order formula is

$$\mathrm{LTE}(h) = \frac{h^{14}y^{(14)}(x)}{242\,464\,622\,400}. \tag{24}$$

In Table 3, we list the absolute errors in $y(x)$ obtained by the new method and some previous ones with $h = \pi/5$. From the table, one can see that the accuracy of the solution for problem (18) has been improved gradually, from firstly 10^{-2} by [4] to about 10^{-9} by our new method. It shows that the new method has the great advantage in accuracy. In order to compare our new method with our previous one [3], we also list the numerical results obtained by these two methods in Table 4. In the table, the first three columns are for the previous method and they are calculated with three different set of parameters: (1) $a = -4$, $b = -0.0017$, which gives the widest interval of periodicity, (2) $a = -2$, $b = -0.00215$, which is in the middle of the stable region, and (3) $a = -0.75$, $b = -0.002$, which gives the highest accuracy. From the table, we can see that the new P-stable four-step Obrechkoff method also can compete with the previous one in accuracy, even for the case in which the parameters are chosen for the highest accuracy.

Problem 4. We consider the Bessel equation:

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{0.25}{x^2}\right)y(x) = 0, \quad y'(1) = 0.0954005, \quad y(1) = 0.671397. \tag{25}$$

Table 4
Absolute errors, $|\Delta y_n| = |y_n - y(nh)|$, calculated for the problem (18), with $h = \pi/2, \pi/5, \pi/8$, by the method [3] and our new method. For method [3], there are three different sets of parameters (a, b): $(a, b)^1$ with $a = -4$, $b = -0.0017$, $(a, b)^2$ with $a = -2$, $b = -0.00215$, $(a, b)^3$ with $a = -0.75$, $b = -0.002$, New for our new method (2)

h	x	$(a, b)^1$	$(a, b)^2$	$(a, b)^3$	New
$\frac{\pi}{2}$	π	0	0	0	0
	2π	$1.11 \cdot 10^{-3}$	$1.39 \cdot 10^{-3}$	$4.85 \cdot 10^{-4}$	$2.29 \cdot 10^{-3}$
	4π	$2.31 \cdot 10^{-4}$	$8.99 \cdot 10^{-3}$	$2.19 \cdot 10^{-3}$	$2.92 \cdot 10^{-3}$
	6π	$1.13 \cdot 10^{-3}$	$1.69 \cdot 10^{-3}$	$1.17 \cdot 10^{-2}$	$1.50 \cdot 10^{-3}$
	8π	$2.52 \cdot 10^{-4}$	$6.00 \cdot 10^{-4}$	$6.34 \cdot 10^{-2}$	$1.37 \cdot 10^{-4}$
	10π	$1.31 \cdot 10^{-3}$	$4.17 \cdot 10^{-3}$	$2.38 \cdot 10^{-1}$	$1.97 \cdot 10^{-4}$
$\frac{\pi}{5}$	π	$6.53 \cdot 10^{-9}$	$3.48 \cdot 10^{-10}$	$7.93 \cdot 10^{-10}$	$5.76 \cdot 10^{-9}$
	2π	$9.87 \cdot 10^{-8}$	$4.58 \cdot 10^{-8}$	$1.00 \cdot 10^{-8}$	$1.44 \cdot 10^{-8}$
	4π	$1.54 \cdot 10^{-8}$	$6.25 \cdot 10^{-9}$	$3.31 \cdot 10^{-9}$	$2.02 \cdot 10^{-10}$
	6π	$5.18 \cdot 10^{-8}$	$2.47 \cdot 10^{-9}$	$4.48 \cdot 10^{-9}$	$1.46 \cdot 10^{-8}$
	8π	$5.12 \cdot 10^{-8}$	$9.97 \cdot 10^{-9}$	$4.28 \cdot 10^{-9}$	$4.78 \cdot 10^{-10}$
	10π	$4.85 \cdot 10^{-8}$	$7.97 \cdot 10^{-9}$	$6.92 \cdot 10^{-9}$	$1.32 \cdot 10^{-8}$
$\frac{\pi}{8}$	π	$9.31 \cdot 10^{-10}$	$1.60 \cdot 10^{-12}$	$1.99 \cdot 10^{-11}$	$5.94 \cdot 10^{-13}$
	2π	$1.72 \cdot 10^{-9}$	$5.68 \cdot 10^{-12}$	$1.62 \cdot 10^{-11}$	$2.11 \cdot 10^{-12}$
	4π	$4.12 \cdot 10^{-10}$	$1.35 \cdot 10^{-11}$	$2.61 \cdot 10^{-11}$	$7.79 \cdot 10^{-12}$
	6π	$1.15 \cdot 10^{-9}$	$1.96 \cdot 10^{-11}$	$1.49 \cdot 10^{-11}$	$1.68 \cdot 10^{-11}$
	8π	$1.27 \cdot 10^{-9}$	$2.32 \cdot 10^{-11}$	$6.43 \cdot 10^{-12}$	$2.88 \cdot 10^{-11}$
	10π	$7.49 \cdot 10^{-10}$	$2.42 \cdot 10^{-11}$	$1.44 \cdot 10^{-11}$	$4.33 \cdot 10^{-11}$

The exact solution for this problem is $y(x) = \sqrt{2/\pi x}\,\sin(x)$. To avoid the singular point at zero, the integration of (25) is from 1. In order to improve the accuracy and efficiency for this special problem, we use the following first-order formula:

$$\begin{aligned}
y'_{n+1} &= \frac{1}{h}\left(\Gamma_0 + hy'_{n-1} + h\left(\frac{3}{56}\Phi + \frac{h^2}{10080}\Omega\right)\right),\\
\Gamma_0 &= y_{n+1} + y_{n-1} - 2y_n,\\
\Phi &= 3(y''_{n+1} + y''_{n-1}) + 50y''_n,\\
\Omega &= 2580y_n^{(4)} + 61h^2 y_n^{(6)}.
\end{aligned} \tag{26}$$

The local truncation error is

$$\mathrm{LTE}(h) = \frac{-97h^{10}y^{(10)}(x)}{50\,803\,200}. \tag{27}$$

In Table 5, we compare the numerical error at $nh = 8.0$ of our new method, with method (3.2) and (3.5) in [7]. From the table, we can see the our method is much more accurate than method (3.2) and nearly the same as (3.5).

Table 5
Absolute errors, $|\Delta y_n| = |y_n - y(nh)|$ at $nh = 8.0$ are listed with $h = 2^{-1}, 2^{-2}, 2^{-3}$ for problem (25) by the methods (3.2), (3.5) in [7] and the new method (2) as in the column of 'New'

h	(3.2)	(3.5)	New
2^{-1}	$6.67 \cdot 10^{-5}$	$2.55 \cdot 10^{-8}$	$1.57 \cdot 10^{-8}$
2^{-2}	$4.43 \cdot 10^{-6}$	$3.88 \cdot 10^{-10}$	$7.94 \cdot 10^{-11}$
2^{-3}	$2.78 \cdot 10^{-7}$	$5.82 \cdot 10^{-12}$	$1.83 \cdot 10^{-12}$

All the computations are carried out by the famous algorithm system *Mathematica5.0* on an IBM PC-AT with AMD Athlon XP 2500+, 512M memory.

4. Conclusion

In this paper, we use trigonometric fitting to extend the interval of periodicity of our previous Obrechkoff four-step method to infinity and to make it a P-stable one. It greatly improves the performance of our previous Obrechkoff four-step method and extends its application range. Our work not only verify the Lambert's linear symmetrical theory that a linear symmetric multi-step can have nonzero interval of periodicity, but also generalise the idea that a linear symmetric multi-step can have infinite interval of periodicity. From the numerical test, we show that our new method is a powerful numerical tool to solve a second-order differential equation.

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